Some Notes for the Hackathon

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1 FRW Equations

Originally Einstein Suggested that his field equations should read

$$R_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1}$$

However, when the covariant derivative is applied to the Ricci curvature tensor we find that $\nabla^{\mu} R_{\mu\nu} \neq 0$. We know that $\nabla^{\mu} T_{\mu\nu} = 0$ infers the conservation of energy and momentum so there is something wrong here. In fact Hilbert figured this out and started adding the $-(1/2)Rg_{\mu\nu}$. There was a mysterious and controversial change in Einstein's expression between a submitted and a refereed version of his paper on General Relativity. Some people accuse him of plagarism, the idea is that having seen Hilbert's correction, he realises and tries to cover his error. I personally have no idea but I think that since he did the hard conceptual work, I can forgive him cheating a little bit at the end in panic to make sure someone else didn't get all the credit, and anyway we will probably never know the full story.

The proper, working field equations read

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2}$$

Where $G_{\mu\nu}$ is called the Einstein tensor $T_{\mu\nu}$ is the stress energy tensor. The cosmological principle also applies to the stress energy in the Universe so that the net momentum is zero. Also the net vorticity of the stuff in the Universe should be zero, or there will be a preferred direction corresponding to the axis around which that vorticity is occurring. So we are left with the time-time component of density ρ and the spatial components which are only pressure, i.e. P and nothing else.

$$T_{\mu\nu} = \rho u_{\mu} u_{\nu} + P \left(g_{\mu\nu} + u_{\mu} u_{\nu} \right) \tag{3}$$

where $u_{\mu} = (1, 0, 0, 0)$ is the 4-velocity of the matter in the frame of the Universe where we are moving parallel to the time coordinate t of the Robertson Walker Metric.

The coordinates r, θ, ϕ drop out which is a good thing if we want out Universe to obey the cosmological principle and we get

$$\begin{aligned} G_{00} &\rightarrow 3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{k}{a^2} = 8\pi G\rho & (4) \\ G_{11} &\rightarrow 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -8\pi G P & (5) \\ G_{00} - 3G_{11} &\rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P\right) & (6) \end{aligned}$$

So the first thing we can see is that $\dot{a}$, i.e. the rate at which the Universe expands or contracts depends upon the density and pressure of stuff in the Universe, the spatial curvature of the Universe and the cosmological constant Λ. The quantity $\dot{a}/a = H$ is called the Hubble rate and its value today H_0 is called the Hubble constant. We will use H and $\dot{a}/a$ interchangeably so be careful.

The age of the Universe today is denoted t_0 and the scale factor today is denoted a_0, the Subscript zero $_0$ means the value of a particular quantity in the Universe *today*).

1.1 Redshift

Photons which are emitted at some early time t_e and arrive today t_0 are stretched because the scale factor of the universe $a_0 > a_e$. This factor is called redshift z

$$1 + z = \frac{a_0}{a_e} \tag{7}$$

You can use z instead of time, $z = 0$ is today, $z = 1$ is when the scale factor is half the size it is today.

2 Energy density in the Universe

so we have seen that $3H^2 = 8\pi G\rho - ka^{-2}$ and we have measured H so we can start to makes some inferences about ρ and k. Of course there may be different kinds of stuff in the Universe so in general

$$\rho_{total} = \rho_x + \rho_y + \rho_z + ... \tag{8}$$

For example, the energy density of matter ρ_M and the energy density of radiation ρ_γ. Conservation of energy tells us that the *covariant derivative* (divergence) of the stress energy tensor should be zero. This actually comes also from the Bianchi identity which is a seperate concept in differential Geometry. Each index of the Stress-Energy tensor can be raised by the metric tensor in the normal way, i.e. $T^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}T_{\alpha\beta}$. The true covariant derivative is expressed as

$$\frac{\nabla T^{\mu\nu}}{\partial x^\nu} = \nabla_\nu T^{\mu\nu} = \partial_\nu T^{\mu\nu} + \Gamma^{\nu}_{\nu\sigma}T^{\sigma\mu} + \Gamma^{\mu}_{\nu\sigma}T^{\nu\sigma} = 0 \rightarrow \dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P) \tag{9}$$

or you can also get this from the first law of thermodynamics, i.e, $dE = -PdV$. Note $TdS = 0$ since the expansion of the Universe is adiabatic normally, although in the Early Universe that can change.

In general an equation of state is a relationship between pressure, density and temperature. In cosmology, the equation of state is often simpler and is caled w and it is defined as the relationship between pressure and density. For cosmology usually we write simply

$$P = w\rho \tag{10}$$

and that is that. We will see later that the equation of state of dark energy *may* be a function of time, or at least people are testing that.

We multiply equation (9) by dt/da and then we get

$$\frac{d\rho}{da} = -\frac{3}{a}(\rho + P) = -\frac{3\rho}{a}(1 + w) \tag{11}$$

which has solution

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)} \tag{12}$$

2.0.1 Matter

The matter energy density ρ_M is due to "normal" dust particles which get diluted like the inverse of the volume as the Universe increases $\rho_M = n * m$ where n is the number density of particles and m is the mass of an individual dust particle.

$$n = \frac{\text{number}}{volume} = \frac{\text{constant}_1}{a^3}$$

$$nm = \frac{\text{number} \times \text{mass}}{volume} = \frac{\text{constant}_2}{a^3}$$

so that $\rho_M \propto a^{-3}$ BUT for any density we know that $\rho \propto a^{-3(1+w)}$ So that for matter $w_M = 0$ in other words pressure $P = 0$ which is what we expect. Baryons are essentially matter (we ignore the electrons since their mass is so much smaller) at least on cosmological scales, as is dark matter (more later)

2.1 Models with spatial curvature

Now recall that

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{13}$$

where $k > 0$ corresponds to a positively curved space, $k = 0$ corresponds to flat space and $k < 0$ corresponds to a negatively curved Universe. Remember some people set $k = -1, 0, 1$ in which case the radius of curvature of the Universe is set by $1/a$ while other people allow k to be a free number because they would like to set the scale factor today equal to one. Writing that in various equivalent ways $a_0 = a(\text{today}) = a(t_0) = a(z = 0) = 1$, in which case the radius of curvature is set by $\sqrt{|k|}$.

For simplicity take $\rho = \rho_M \propto a^{-3}$ then equation (13) becomes

$$\dot{a}^2 = \frac{8\pi G}{3}\frac{\rho_0 a_0^3}{a} - k \tag{14}$$

- If $k = 0$ then as $a \to \infty$, $\dot{a} \to 0$ and the universe grinds to a halt at $t = \infty$.
- If $k < 0$, $\dot{a}$ never goes to zero and the Universe expands forever.
- If $k > 0$ then $\dot{a}$ becomes zero at a finite time. Now $3\ddot{a} = -4\pi G\rho a/3$ is always negative so if $k > 0$ the Universe collapses after a finite time $\to$ BIG CRUNCH

2.2 ρ_{crit} and Ω

This bit is super easy so long as you actually read it!

The Friedman equations look like

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \equiv \frac{8\pi G}{3}\rho_{crit} \tag{15}$$

So that if $k = 0$ then $\rho = \rho_M + \rho_\gamma + \rho_X + ... = \rho_{crit}$ or to put it another way, for each value of the Hubble rate H, there is is a density ρ_{crit} which corresponds to a spatially flat Universe $k = 0$

$$\rho_{crit} = \frac{3H^2}{8\pi G} \tag{16}$$

Now the Ω parameters are simply rescaled measurements of the density, measured in terms of the critical density, i.e. $\Omega_M = \rho_M/\rho_{crit}$, $\Omega_\Lambda = \rho_\Lambda/\rho_{crit}$ and more generally $\Omega_X = \rho_X/\rho_{crit}$.

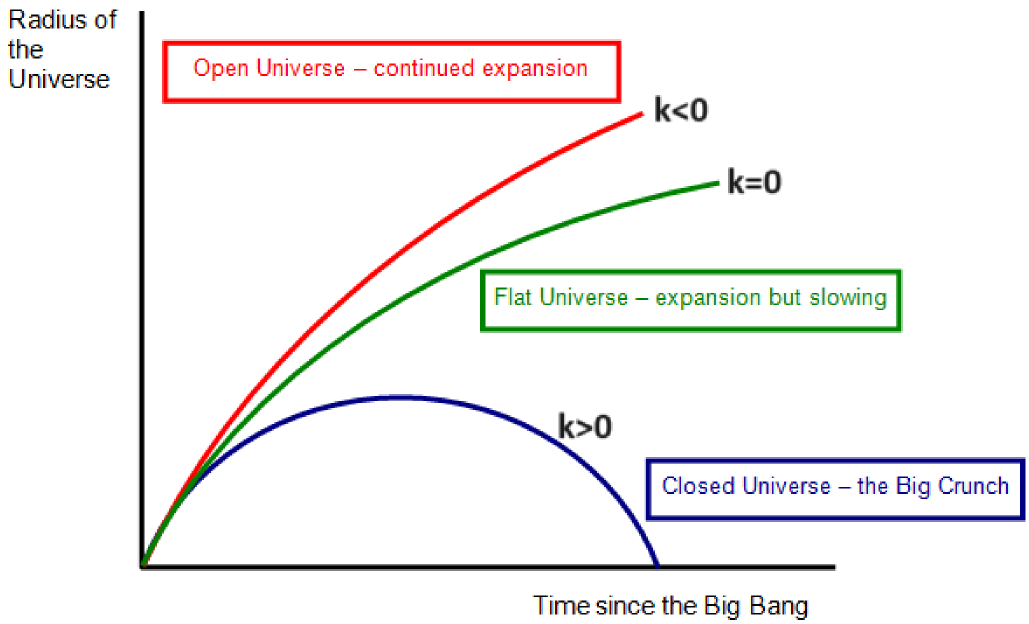

Figure 1: Different fates for different curvatures

Example

Now consider an open Universe $k < 0$ which means that $\rho < \rho_{crit}$. Lets say that today ρ_{M0} is 50% ρ_{crit0} and $\rho_{\gamma 0}$ is 10% ρ_{crit0} then

$$\Omega_0 = \Omega_{M0} + \Omega_{\gamma 0} = 0.5 + 0.1 = 0.6 \tag{17}$$

Note, we are referring to these values of ρ and Ω *today*, hence the $_0$ subscript.

Curvature written as Density

Sometimes it is convenient to treat curvature as a seperate source of density that we call Ω_k, such that

$$\Omega_M + \Omega_\gamma + \Omega_x + \Omega_y + ... + \Omega_k = 1 \tag{18}$$

then in this case all of the Ω_i (so long as you include Ω_k) will always sum to unity (i.e. 1) by definition and

$$\Omega_k = -\frac{k}{a^2 H^2} \tag{19}$$

3 The Cosmological Constant

"The term is necessary only for the purpose of making possible a quasi-static distribution of matter, as required by the fact of the small velocities of the stars" (Einstein 1917).

So the field equations are

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{20}$$

which is the simplest combination of these quantities which has a zero covariant derivative. However, the constraint $\nabla^{\mu}(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) = 0$ of course has room for a constant of integration, so it can be written

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{21}$$

Einstein introduced this constant Λ and called it the "cosmological constant" in 1917. This was before the Friedman-Lemaitre-Robertson-Walker equations which came only later in 1922-24. Einstein realised very early on that his equations would lead to a dynamical Universe which would follow an expanding or contracting trajectory. He didn't like this conceptually and tried to stop the Universe from expanding using this cosmological constant.

3.1 Cosmological constant in FRW equations

It is easier to see how Λ can stop expansion using the FLRW equations.

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} \tag{22}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3} \tag{23}$$

One can assume that $\rho = \rho_M$ i.e. a matter dominated Universe with $P = 0$ then one can set

$$\Lambda = 4\pi G \rho_M \qquad \rightarrow \qquad \ddot{a} = 0 \tag{24}$$

and then we see that the following choice for the curvature

$$\Lambda = 4\pi G \rho_M \quad \& \quad k = 4\pi G \rho_M a^2 \qquad \rightarrow \qquad \dot{a} = 0 \tag{25}$$

so that by setting the cosmological constant and the curvature to suitable values, there exists a solution where the expansion of the Universe is zero and the Universe remains static.

This solution, although valid, is unstable, since if ρ_M is just a little bit too big or too small, the Universe runs away from this fixed point. So this was an extremely fine tuned solution, not stable to perturbations. Einstein knew this, everyone knew this and no-one could come up with anything more sensible to stop the Universe from expanding. Then in the 1920s Hubble discovered that the

Universe *was* actually expanding. Einstein therefore rejected the cosmological constant and later told Gamow that it was his *"biggest blunder"*. It was not until the mid 1990s that people realised that getting rid of this term may have been premature.

3.2 Cosmological Constant as energy density

One can treat the cosmological constant as being just that - a cosmological constant, a *deus ex machina* term on the geometry side of Einstein's equations which comes from no-where with no obvious origin, or one can try and understand where the heck it may have come from.

We can write the cosmological constant as a form of energy density by moving it from the left, *geometry* side of the Einstein's equations to the right *energy/stuff* side.

$$\begin{aligned} \rho_{eff} &= \rho + \frac{\Lambda}{8\pi G} = \rho + \rho_\Lambda \\ P_{eff} &= P - \frac{\Lambda}{8\pi G} = P - P_\Lambda \end{aligned} \tag{26}$$

so that Λ acts like an energy density with negative pressure. We can write the relationship between pressure and density as

$$\begin{aligned} P &= w\rho \\ P_\Lambda &= w_\Lambda \rho_\Lambda = -\rho_\Lambda \\ w_\Lambda &= -1 \end{aligned} \tag{27}$$

where w_X is the equation of state of energy density X.

So lets now set $k = 0$ and set $\rho = 0$ but set $\rho_\Lambda \neq 0$ then

$$G_{00} \to 3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G \rho_\Lambda \tag{28}$$

which has solution

$$a(t) = a_0 \exp\left(\sqrt{\frac{8\pi G \rho_\Lambda}{3}} t\right) = a_0 e^{Ht} \tag{29}$$

so that Universe expands exponentially! *Inflationary* expansion. Best theory is that this happened in the early Universe and possibly starting again now. MORE LATER. ρ_Λ therefore does not go down as the Universe expands, very unusual, although it does happen sometimes in quantum field theory. The energy density of normal matter gets diluted as the Universe expands.

If we *do* assume that we have a cosmological constant with an energy density ρ_Λ then we can see that as we go backwards in time to higher redshifts, the ratio between dark energy and all other forms of matter (i.e. dark matter and baryons) is given by

$$\frac{\rho_\Lambda}{\rho_M} = \frac{\Omega_{\Lambda 0}}{\Omega_{M0}(1+z)^3} \tag{30}$$

such that we can define a redshift where the densities are equal to each other.

Observations

4 Hubble Law and Hubble constant

As the Universe expands, galaxies move away from each other because $a(t)$ increases. The further they are from us the faster they appear to move.

If we consider a galaxy at co-moving distance r. At time t it is a distance $ra(t)$ away whereas at time t_0 it is a distance $ra(t_0)$ away so its velocity from us it

$$v_{hub} = r\frac{a(t_0) - a(t)}{t_0 - t} \tag{31}$$

Now for low redshift objects, $a(t) \sim a(t_0)$ and we can expand $a(t)$ in terms of $a(t_0)$

$$\begin{aligned} a(t) &= a(t_0) + \left.\frac{da}{dt}\right|_{t_0} (t - t_0) + \frac{1}{2}\left.\frac{d^2a}{dt^2}\right|_{t_0} (t - t_0)^2 + ... \\ &= a(t_0)\left[1 + H_0\,(t - t_0) - \frac{1}{2}q_0 H_0^2\,(t - t_0)^2 + ...\right] \end{aligned} \tag{32}$$

where q_0 is known as the deceleration parameter

$$q_0 = -\frac{\ddot{a}}{aH_0^2} = -\frac{4\pi G}{3H_0^2}\,(\rho + 3P) \tag{33}$$

The deceleration parameter used to be very important as the only observations we had of the Universe were at low redshift. With the development of better telescopes, nobody really talks about q_0 any more because now we can go to much higher redshifts and the pertubative expansion isn't really interesting any more, apart from to obtain the low redshift Hubble law. For small redshifts $H_0(t - t_0)$ is very small so only the first two terms are important.

$$v_{hub} = r\frac{a(t_0) - a(t)}{t_0 - t} = -r\frac{a(t_0)H_0\,(t - t_0)}{t_0 - t} = ra(t_0)H_0 = H_0 d \tag{34}$$

where d is the distance. Also

$$z \simeq \frac{H_0 d}{c} \tag{35}$$

5 Measuring the Hubble Constant at Low Redshift

5.0.1 Magnitudes

Astronomers measure brightness of objects in magnitudes. These are backward and logarithmic.

Object	m
Sun	-27
full moon	-12
venus	-5
Vega (star)	0
Andromeda Galaxy	+4.5
Neptune	+7

The values are apparent magnitudes m - related to brightness we see on earth. We can define the absolute magnitude M which is the magnitude the object would appear if it were 10 pc away.

$$m - M = 5 \log d - 5 \tag{36}$$

where d is the distance in parsecs. If you have m and M you can get d. The Luminosity if the total outflow of energy of object (star etc) per second.

$$\log \left(\frac{L}{L_\odot} \right) = 0.4 \left(M_\odot - M \right) \tag{37}$$

where $L_\odot$ is the luminosity of the Sun and $M_\odot$ is the absolute magnitude of the Sun.

5.0.2 Standard Candles

Standard Candles are astronomical objects we know the absolute magnitude (luminosity) of. E.g. Cepheid variables - stars which pulsate. The frequency of their pulsations is a direct function of their luminosity.

1. measure period of cepheid - calculate luminosity
2. measure apparent magnitude m with telescope $m - M \to d$
3. measure redshift of spectral lines in host galaxy (galaxy where Cepheid lives)
4. draw Hubble diagram

Another type of standard candles are type 1a supernovae (Sn1a). Sn1a are thought to occur in binary systems when big stars pour matter onto small stars. degenerate white dwarf core develops inside the small star. When the core mass grows to $M_{Ch} \simeq 1.4 M_\odot$ i.e. the Chandrasekhar mass they explode and all have very similar magnitudes.

Type 1a supernovae are quite rare but can be observed at very large redshifts, $z > 1$.

Cepheid variables are more common but not observed at large distances (not bright enough). Cepheids are then used to obtain H_0.

Cepheid variable observations take place in galaxies which also harbour type 1a standard candles, which allow us to calibrate the distance ladder.

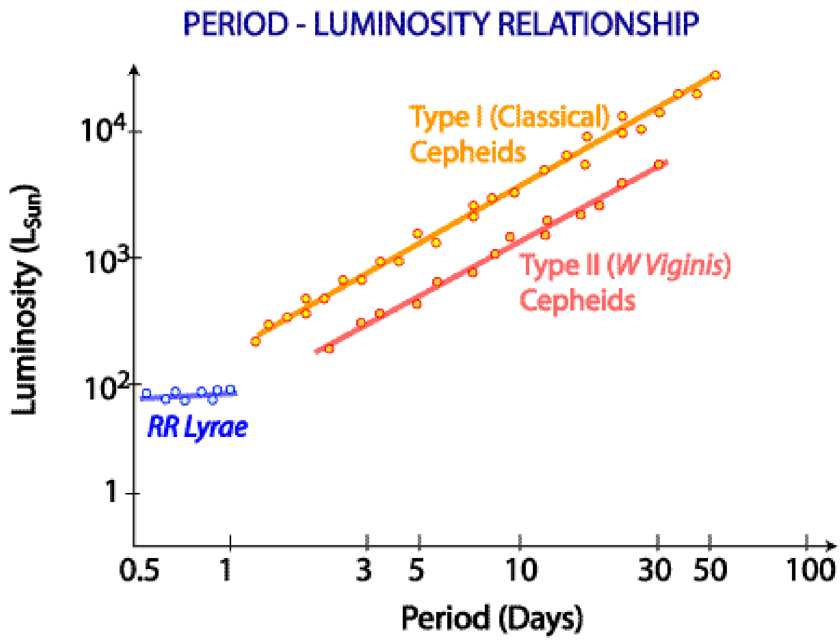

Figure 2: Measuring the Hubble Rate Using Cepheids

5.0.3 Measured value of the Hubble Constant.

The Hubble constant has a value of the order of 100 $\mathrm{kms}^{-1}\mathrm{Mpc}^{-1}$

$$H = h \times 100 \mathrm{kms}^{-1}\mathrm{Mpc}^{-1} \tag{38}$$

and the measured value of h is approximately $h = 0.65 - 0.75$ using equations such as (35), we simply compare m and M to get d and compare that to z. If we want to measure the evolution of the Hubble constant properly over the history of the Universe we need to take into account that d changes due to the evolution of the scale factor. More later.

6 Measuring the Hubble rate at High Redshift

If we want to see how the Hubble rate has changed over the history of the Universe, we need to look at larger distances and therefore larger redshifts. In those situations, the expansion used before is not enough and we need to look at the luminosity distance.

6.0.1 Luminosity Distance

If a supernova at comoving distance r sends out a photon at time t_{emit} the fraction we detect at time t_{obs} with a telescope of area A is

$$fraction = \frac{A}{4\pi\left(a(t_{obs})r\right)^2} \tag{39}$$

BUT each photon is redshifted so that

$$E_\gamma(t_{obs}) = \frac{E_\gamma(t_{emit})}{1+z} \tag{40}$$

And the time between arrival of photons Δt is also stretched by a factor $(1+z)$. The flux F, which is the energy per unit area per second which arrives at the telescope is given by.

$$F = \frac{L}{4\pi a^2(t_0)r^2(1+z)^2} \tag{41}$$

where L is the luminosity, the total energy emitted by the source per second. In a flat space-time, $F = L/(4\pi d^2)$ where d is the distance so in an FRW universe with curved space and/or space-time we call $d_L = a(t_0)r(1+z)$ the Luminosity distance.

To fit the Hubble diagram and determine $\Omega_M, \Omega_\Lambda, \Omega_K$ etc., we need the exact expression for the luminosity distance $d_L(z)$. To get this, we need the relationship between r and z, in other words, photons we see now emitted at a redshift z come from what comoving distance r? To do this, first we establish the relationship between the comoving position of the source r and the time the photons were emitted t and then we find a relationship between t and z.

Photons take a light like trajectory, i.e. $ds^2 = 0$, and radial photons that arrive here do not change their value of θ or ϕ so we can write

$$ds^2 = 0 \rightarrow \frac{dr}{dt} = \frac{\sqrt{1-kr^2}}{a(t)} = \frac{\sqrt{1-kr^2}(1+z)}{a_0} \tag{42}$$

so that

$$\frac{a_0 dr}{\sqrt{1-kr^2}} = (1+z)dt = (1+z)dz\frac{dt}{dz} = -\frac{dz}{H(z)} \tag{43}$$

$$\int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = \begin{cases} \frac{\arcsin(r\sqrt{k})}{\sqrt{k}} & k > 0 \\ r & k = 0 \\ \frac{\text{arcsinh}(r\sqrt{-k})}{\sqrt{-k}} & k < 0 \end{cases} \tag{44}$$

which means we can then write the luminosity distance as

$$d_L = a(t_0)(1+z)r = \frac{c(1+z)}{H_0\sqrt{|\Omega_K|}}\mathcal{S}\left[\sqrt{|\Omega_K|}\int_0^z \frac{dz'}{\tilde{H}(z')}\right] \tag{45}$$

where $\tilde{H} = H/H_0$ and

$$\mathcal{S}[x] = \begin{cases} \sin(x) & k > 0 \\ x & k = 0 \\ \sinh(x) & k < 0 \end{cases} \tag{46}$$

$$d_L = a(t_0)(1+z)r = \frac{c(1+z)}{H_0} \int_0^z \frac{dz'}{\tilde{H}(z')} \tag{47}$$

We then use this luminosity distance to work out the difference between the apparent and absolute magnitude. Using the luminosity distance (45) we are able to fit the type 1a supernovae luminosity-redshift relationship. A flat $\Omega_M = 1$ Universe without a cosmological constant is disfavoured, the data actually is difficult to reconcile with a Universe with no cosmological constant. This means that the Universe is accelerating. It also means that if the stuff really is a cosmological constant, it only became important recently - this is a problem.

7 The Problem

$$\mu = m - M$$

$$\mu = 5\log_{10}\left(\frac{d_L}{10\,\mathrm{pc}}\right)$$

Equivalently, if d_L is measured in Mpc,

$$\mu = 5\log_{10}\left(\frac{d_L}{\mathrm{Mpc}}\right) + 25.$$

For a flat ΛCDM universe,

$$d_L(z) = (1+z)\,\frac{c}{H_0}\int_0^z \frac{dz'}{E(z')},$$

where

$$E(z) = \frac{H(z)}{H_0} = \sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda}.$$

For a flat universe,

$$\Omega_\Lambda = 1 - \Omega_m.$$

So the model prediction is

$$\mu_{\mathrm{model}}(z) = 5\log_{10}\left[\frac{1}{\mathrm{Mpc}}(1+z)\frac{c}{H_0}\int_0^z \frac{dz'}{\sqrt{\Omega_m(1+z')^3 + \Omega_\Lambda}}\right] + 25.$$

The chi-squared to minimise is

$$\chi^2 = \sum_i \frac{\left[\mu_i - \mu_{\mathrm{model}}(z_i)\right]^2}{\sigma_{\mu,i}^2}.$$

In the data I have given you, they have assumed one particular absolute magnitude to obtain μ which might be wrong. Also, we don't know precisely what H_0 is, include an arbitrary offset $\mathcal{M}$:

$$\mu_{\mathrm{model}}(z) = 5\log_{10}\left[\frac{1}{\mathrm{Mpc}}(1+z)\frac{c}{H_0}\int_0^z \frac{dz'}{\sqrt{\Omega_m(1+z')^3 + \Omega_\Lambda}}\right] + 25 + \mathcal{M},$$

we treat $\mathcal{M}$ as a nuisance parameter - we have to vary it to find the lowest minimum χ^2 possible for each value of Ω_Λ